A STATE SPACE APPROACH TO NON-INTERACTING CONTROLS

G. BASILE and G. MARRO

Abstract. The well-known problem of synthesizing non-interacting controllers for linear time-invariant systems is considered.

A particular approach, based on properties of linear transformations related to the state variable description of the system, is developed, avoiding functional transforms and transfer-function matrices.

Making use of some recent results in linear system theory and notations and concepts introduced by the authors in previous papers, necessary and sufficient conditions are derived for obtaining an asymptotically stable control system, noninteracting on given subspaces of the output space.

1. Introduction

More than ten years ago the problem of synthesizing non-interacting and invariant controllers was broadly investigated. The early approaches, based on the use of transfer-functions matrices, gave apparently complete results, which, nevertheless, were very impractical to handle, especially for high order systems.

Following the modern trend of studying control problems in the state space, for an easier use of automatic computation facilities, recently some Authors [1-5] have reformulated the non-interaction problem in terms of state variables, also giving it a new name: «decoupling». In their contributions controllers based on algebraic state feedback are employed for obtaining the non-interaction by means of a suitable modification of the structure of the system. Such approach is not general and exhaustive because dynamic systems often can be decoupled only using dynamic controllers, instead of purely algebraic ones: a simple example reported in section 5 will clarify this important assertion.

The results obtained in this paper are:

1) a set of necessary conditions for the non-interaction, which state precise requirements on the structure of the controlled system;
2) the proof of the sufficiency of these conditions, assuming a suitable structure for the controller;
3) the outline of a synthesis procedure for non-interacting controllers corresponding to asymptotically stable overall systems.

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The approach is based on properties of controlled invariant subspaces, previously
defined by the same authors [6, 7]. The developed theory is easily understandable
because of its simple formulation and intuitive geometric meaning and is a natural
complement of a similar approach to the problem of state space synthesis of distur-
rance-invariant controllers [8, 9].

2. Statement of the problem

Consider a linear time-invariant dynamical system represented by

\[ \dot{x} = Ax + Bu \]  \hspace{1cm} (2.1)
\[ y = Cx \]  \hspace{1cm} (2.2)

where \( x \in \mathbb{R}^n \) denotes the state, \( u \in \mathbb{R}^m \) the control, \( y \in \mathbb{R}^r \) the output.

The problem this paper is concerned with is the synthesis of an asymptotically
stable non-interacting controller, having the structure shown in Fig. 1. It will be
pointed out in the sequel that such a structure allows to achieve non-interaction with
minimal loss of controllability.

The manipulated input of the controller is denoted by \( r \) and is partitioned in \( p \)
vectors \( r_1, \ldots, r_p \), which correspond to non-interacting actions. Another input is
a feedback connection from the output of the controlled system; of course, the out-
put of the controller coincides with the input of the controlled system. Note that,
while the controlled system is assumed to be purely dynamical, in the controller a
direct or algebraic input-output action is allowed.

The controller state is denoted by \( z \in \mathbb{R}^l \). In the synthesis the integer \( l \) and the
matrices \( A_i, B_i, C_i, D_i, D_o \) are completely free.

For a better understanding of the following exposition it is first necessary to state
in precise terms what is intended for non-interaction.

\[ r = [r_1, \ldots, r_p] \]
\[ u = Ax + Bu \]
\[ y = Cx \]

**Fig. 1 - The non-interacting control apparatus.**

**DEFINITION 2.1:** Given a set of subspaces \( \mathbb{Y}_1, \ldots, \mathbb{Y}_q \) of the output space \( \mathbb{R}^r \),
which intersect only at the origin, the controller shown in Fig. 1 is said non-interacting
on them if, for every \( i \), starting from a rest condition and varying in any way only the
input $\mathbf{r}$, an output trajectory, which completely belongs to the subspace $\mathcal{Y}$, is obtained.

It is always possible to synthesize a controller which realizes the non-interaction or any given set of subspaces, but, of course, it is desirable also to achieve a good controllability on every one of them by means of the corresponding input.

To be precise, if the word "controllability" is intended in the usual sense of dynamic controllability, it is desirable that, while non-interaction is preserved, every input $\mathbf{r}$ allows to reach the maximal subspace contained in the corresponding $\mathcal{Y}$; if, on the other hand, perfect controllability [10] is requested, it is desirable to reach the maximal subspace of perfect output controllability contained in $\mathcal{Y}$. It is clear that in general the dimension of the input vector $\mathbf{r}$ necessary for perfect controllability is greater than for simple dynamic controllability.

In next section a structural analysis for determining such maximal subspaces of controllability is performed and it is proved that they are independent, so that it is not possible to have a better control on one particular of them making worse in some of the other ones.

After, it is shown that, if the system (2.1, 2.2) is completely controllable and observable, it is always possible to synthesize a controller which realizes the non-interaction on the greatest subspaces and such that the overall system (controller and controlled system) is asymptotically stable.

It is interesting to note that, performing the synthesis of the controller, two aims are considered: the non-interaction on a given set of subspaces and the best degree of controllability on every one of them. In the approach discussed here, when these aims are conflicting, controllability is traded for non-interaction.

However, by virtue of its great flexibility, the synthesis procedure can be easily modified to treat also non-interaction problems stated in a different way, as for example the hierarchic non-interaction problem discussed in Section 6.

Let $\mathbf{x} = [x, z] \in \mathbb{R}^{m+1}$ be an augmented state vector, which accounts also for the state variables of the controller. Equations (2.1, 2.2), together with the equations of the controller, can be written in the compact form

$$\dot{x} = A\mathbf{x} + B\mathbf{r}$$

$$y = C\mathbf{x}$$

where the matrices $A$, $B$, $C$ can be easily computed in terms of the matrices $A$, $B$, $B_1$, $B_2$, $C_1$, $D_1$, $D_2$. Note that the equations of the overall system are of the same type as those of the sole controlled system.

Referring to equations (2.3, 2.4), it is possible to state a simple necessary and sufficient condition for non-interaction, given in the following assertion.

**Assertion 2.2**: The system (2.3, 2.4) is non-interacting on the subspaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_p$ if, and only if, the following condition holds
\( \mathcal{C} m \mathfrak{I} (\mathbf{A}, \mathcal{R}(\mathbf{B})) \subseteq \mathcal{Y}_i \quad (i = 1, \ldots, p) \quad (*) \) (2.5)

where \( \mathbf{B}_i \) \((i = 1, \ldots, p)\) is the partition of the columns of the matrix \( \mathbf{B} \) corresponding to the partition \( r_i \) \((i = 1, \ldots, p)\) of the input vector \( r \).

The meaning of condition (2.5) is quite manifest: the output space mapping of the subspace \( m \mathfrak{I} (\mathbf{A}, \mathcal{R}(\mathbf{B})) \), which is the reachable set, in the augmented state space, corresponding to the input \( r \), must be contained in \( \mathcal{Y}_i \).

It is also obvious that, when the equality sign in (2.5) holds, every subspace \( \mathcal{Y}_i \) is completely reachable by means of the control vector \( r \).

3. Non-interaction and controlled invariants

Consider the sole controlled system (2.1, 2.2). The following theorem will be proved.

**THEOREM 3.1:** For every output subspace \( \mathcal{Y}_i \) \((i = 1, \ldots, p)\), the greatest subspace \( \mathcal{Y}_i' \subseteq \mathcal{Y}_i \) which can be reached by means of trajectories completely belonging to \( \mathcal{Y}_i \) is defined by the relationships

\[
\begin{align*}
\mathcal{F}_{0i} &= \text{MIC} (\mathbf{A}, \mathcal{R}(\mathbf{B}), \mathbf{C}^{-1} \mathcal{Y}_i) \\
\mathcal{Y}_i' &= \text{mi} (\mathbf{A} + \mathbf{B} \mathbf{H}, \mathcal{F}_{0i} \cap \mathcal{R}(\mathbf{B})) \\
\mathcal{Y}_i' &= \mathbf{C} \mathcal{F}_i'
\end{align*}
\]

(3.1) (3.2) (3.3)

where \( \mathbf{H} \) is any matrix such that \((\mathbf{A} + \mathbf{B} \mathbf{H}) \mathcal{F}_{0i} \subseteq \mathcal{F}_{0i} \).

**Proof:** If, by means of a proper action on the control \( u \), an output trajectory is followed which belongs to the subspace \( \mathcal{Y}_i \), the corresponding state trajectory must belong to the greatest controlled invariant contained in the subspace \( \mathbf{C}^{-1} \mathcal{Y}_i \) [6], i.e. to the subspace (3.1).

As it has been shown in [7], a controlled invariant can be always transformed in a simple invariant by means of a proper state variable feedback; in other words, since the subspace \( \mathcal{F}_{0i} \) satisfies the condition \( \mathbf{A} \mathcal{F}_{0i} \subseteq \mathcal{F}_{0i} + \mathcal{R}(\mathbf{B}) \), there exists at least a matrix \( \mathbf{H} \) such that \((\mathbf{A} + \mathbf{B} \mathbf{H}) \mathcal{F}_{0i} \subseteq \mathcal{F}_{0i} \).

It can be easily proved that the greatest subspace \( \mathcal{Y}_i' \) which can be reached by proper control actions, starting from the origin and following trajectories completely belonging to the controlled invariant \( \mathcal{F}_{0i} \), is given by relationship (3.2): in fact, in any point \( \mathbf{x}^* \in \mathcal{F}_i' \) the set of all control actions which correspond to velocities belonging to \( \mathcal{F}_{0i} \) is \( \mathbf{B} \mathbf{H} \mathbf{x}^* + \mathcal{F}_{0i} \cap \mathcal{R}(\mathbf{B}) \) and, as it has been shown in [7], does not depend on the particular choice of \( \mathbf{H} \). Since these velocities must belong to \( \mathcal{F}_i' \) for every \( \mathbf{x}^* \in \mathcal{F}_i' \), it follows that \((\mathbf{A} + \mathbf{B} \mathbf{H}) \mathcal{F}_i' + \mathcal{F}_{0i} \cap \mathcal{R}(\mathbf{B}) \subseteq \mathcal{F}_i' \), then \( \mathcal{F}_i' \) is an invariant under \( \mathbf{A} + \mathbf{B} \mathbf{H} \) and contains \( \mathcal{F}_{0i} \cap \mathcal{R}(\mathbf{B}) \). It is the least invariant

(*) The symbols employed in equation (2.5) and in subsequent formulas are explained in the Appendix.
under $A + BH$ containing $\mathcal{S}_{st} \cap \mathcal{S}(B)$ because the existence of any other of such invariants contained in it would contradict the reachability of every point of $\mathcal{F}_i$ by means of trajectories belonging to $\mathcal{F}_{st}$.

Relationship (3.3) gives finally, as the mapping of $\mathcal{F}_i$ by the matrix $C$, the subspace $\mathcal{Y}_i$ defined in the statement of the theorem.

Every subspace $\mathcal{F}_i$ (i = 1, ..., p), defined in Theorem 3.1, is clearly the locus of the state trajectories corresponding to motions on $\mathcal{Y}_i$.

The output on every $\mathcal{Y}_i$ (i = 1, ..., p) can be controlled completely and independently only if $\mathcal{Y}_i = \mathcal{Y}_a$ as the following corollary states.

**COROLLARY 3.2:** For the non-interaction and complete controllability on the subspaces $\mathcal{Y}_i$ (i = 1, ..., p), the condition $\mathcal{Y}_i = \mathcal{Y}_a$ is necessary.

By a similar reasoning, if perfect controllability on every $\mathcal{Y}_i$ (or on a particular $\mathcal{Y}_i$) is required, it can be checked whether the structure of the system is favorable, determining the greatest subspace $\mathcal{Y}_i^{(k)} = \mathcal{Y}_a$, where the perfect output controllability with respect to the time derivatives of an arbitrary order $k$ can be obtained. Computational procedures for the solution of this problem have been given in [15]; they provide also the corresponding locus of state trajectories $\mathcal{F}_i^{(k)}$. Clearly, in order to achieve contemporarily the non-interaction and complete perfect controllability on the subspace $\mathcal{Y}_a$, the necessary condition $\mathcal{Y}_i^{(k)} = \mathcal{Y}_a$ has to be satisfied.

In conclusion, a tool has been given for the structural analysis of the controlled system, in order to deduce information about the possibility of decoupling and controlling the outputs in any given way.

4. Proof of the sufficiency of the derived necessary conditions

It is shown in this section that an asymptotically stable system having the structure illustrated in Fig. 1, performing a complete dynamic non-interacting control on the subspaces $\mathcal{Y}_i^{(1)}, ..., \mathcal{Y}_i^{(p)}$ defined in Theorem 3.1 can always be synthesized. In this way the sufficiency of the condition stated in Corollary 3.2 will be proved. A similar procedure can be applied also to the non-interacting subspaces of perfect output controllability $\mathcal{Y}_i^{(1)}, ..., \mathcal{Y}_i^{(p)}$, previously defined, in order to obtain a complete perfect non-interacting controller.

**THEOREM 4.1:** If the system (2.1, 2.2) is completely controllable and observable, it is possible to synthesize an asymptotically stable (*) control system of the type shown in Fig. 1 which performs a complete dynamic non-interacting control on every subspace $\mathcal{Y}_i$ (i = 1, ..., p), defined in the statement of Theorem 3.1.

Proof: The matrix $A$ is first assumed to be asymptotically stable. Later this hypothesis will be removed; nevertheless, it is actually true for many controlled systems.

(*) To be precise, it is shown in the proof that the eigenvalues of the matrix $\hat{A}$ of equation (2.3) are completely arbitrary.
For the sake of notational simplicity, the particular case when only two subspaces of non-interaction are given is considered. Let denote them and the corresponding loci of state space trajectories by \( \mathcal{F}_1', \mathcal{F}_2', \mathcal{F}_1, \mathcal{F}_2 \) respectively.

Being \( \mathcal{F}_1', \mathcal{F}_2' \) controlled invariants, there exist two matrices \( H_1, H_2 \) such that
\[
(A + BH_1) \mathcal{F}_1' \subseteq \mathcal{F}_1', \quad (A + BH_2) \mathcal{F}_2' \subseteq \mathcal{F}_2'.
\]

In the augmented state space of the vectors \( \mathbf{x} = [x, z] = [x_1, z_1, z_2] \), where \( z_1 \) and \( z_2 \) have the same dimension as \( x \), define the matrix
\[
\hat{\Lambda} = \begin{bmatrix}
A & BH_1 & BH_2 \\
O & A + BH_1 & O \\
O & O & A + BH_2
\end{bmatrix}
\tag{4.1}
\]

The subspaces
\[
\mathcal{F}_1' = \{ \mathbf{x} : \mathbf{x} \in \mathcal{F}_1', \ z_1 = x, \ z_2 = 0 \}
\tag{4.2}
\]
\[
\mathcal{F}_2' = \{ \mathbf{x} : \mathbf{x} \in \mathcal{F}_2', \ z_1 = 0, \ z_2 = x \}
\tag{4.3}
\]
are clearly invariant under \( \hat{\Lambda} \).

Then, define
\[
\mathbf{B} = \begin{bmatrix}
G_1 & G_2 \\
G_1 & O \\
O & G_2
\end{bmatrix}
\quad \mathbf{C} = [\mathbf{C} \ O \ O],
\tag{4.4}
\]
where the partition by columns corresponds to the partition of the input \( r \) in two non-interacting inputs \( r_1, r_2 \) and the matrices \( G_1, G_2 \) are such that
\[
\mathcal{R}(G_1) = \mathcal{F}_1' \cap \mathcal{F}(B) \quad (i = 1, 2)
\tag{4.5}
\]
It is a very simple matter to verify that the controller so defined is compatible with the structure illustrated in Fig. 1. Furthermore, it is non-interacting and complete on every \( \mathcal{F}_i' \), because it satisfies relationship (2.5) with the equality sign.

The eigenvalues of the matrix \( \hat{\Lambda} \) coincide with those of \( A, A + BH_1, A + BH_2 \). Stability has been temporarily assumed for \( A \), but is not assured for \( A + BH_1, A + BH_2 \).

However, being the system (2.1, 2.2) completely controllable, two matrices \( H_1, H_2 \), \( H_1 \) can be determined such that \( A + B (H_1 + H_2) \) have arbitrary eigenvalues [11, 12] and the completely reachable invariants \( \mathcal{F}_1' \) and \( \mathcal{F}_2' \) are preserved [13].

It is sufficient now to substitute in (4.1) \( H_1 + H_2, H_1 + H_2 \) for \( H_1, H_2 \) in order to obtain a controller which is non-interacting on \( \mathcal{F}_1', \mathcal{F}_2', \) asymptotically stable and complete (*).

(*) The controller so synthesized has, in general, a greater than necessary state space dimension. It is easily verified that \( z_1 \) and \( z_2 \) can be also assumed to have at the most as many components as the dimensions of \( \mathcal{F}_1' \) and \( \mathcal{F}_2' \) are. In fact, in order to make a controlled invariant to be a simple invariant, a partial state variable feedback, in which only the components of the state with respect to a basis on the controlled invariant are needed, can be used.
Now the hypothesis that $A$ is asymptotically stable is removed. Being the system (2.1, 2.2) completely controllable, there exists a matrix $H$ such that $A + BH$ has arbitrary eigenvalues. Since state variable feedback is not possible, but, on the other hand, the system (2.1, 2.2) is completely observable, by means of a proper observer of the Luenberger type [14] with arbitrary eigenvalues, an estimate of the state which can replace the state in the feedback connection is obtained. The eigenvalues of the overall system are arbitrary, being the union of those of $A + BH$ and those of the observer [15].

Denoting by $x_0 = [x, x_o]$ the corresponding augmented state, where $x_o$ is the state of the observer, it is clear that the subspaces

$$\mathcal{F}_{a_1} = \{x_o : x \in \mathcal{F}_{a_1}\}$$

(4.6)

$$\mathcal{F}_{a_2} = \{x_o : x \in \mathcal{F}_{a_2}\}$$

(4.7)

are controlled invariants whose projections on the state space of the sole controlled system are completely reachable.

A non-interacting controller of the previously described type can be applied to the stabilized system, obtaining again a total controller which has the assumed structure. $\lhd$

5. An illustrative example

In this section it will be shown by a simple example that the non-interaction can be achieved by means of a dynamic controller in cases where algebraic controllers based on state variable feedback are ineffective.

Consider the following particularization of a two-inputs, two-outputs, asymptotically stable linear controlled system:

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 \end{bmatrix}.$$ (5.1)

Recall that the necessary and sufficient condition for decoupling by means of an algebraic controller based on state variable feedback, as given by Falb and Wonham in [3], is expressed by

$$\det \begin{bmatrix} CA^t & B \\ CA & A^t \end{bmatrix} \neq 0,$$ (5.2)

where $C_i$ and $C_j$ are $1 \times 3$ submatrices corresponding to the row-wise partition of $C$ and $d_i, e_i$ the least integers, including zero, such that $C_i A^t B \neq 0 (i = 1, 2)$.

In the considered case, being inequality (5.2) not satisfied, it is not possible to decouple by state variable feedback. The non-interaction can be achieved, on the other
hand, by means of a dynamic controller fitting the general structure represented in Fig. 1. The differential equations of the overall system are specified in Fig. 2.

\[ \begin{align*}
\dot{x}_1 &= -2x_1 + u_2 \\
x_2 &= 2x_1 + x_3 + u_2 \\
x_3 &= x_1 + x_2 + u_1 \\
y_1 &= x_1 \\
y_2 &= x_2 \\
\end{align*} \]

Fig. 2 - A particular case of non-interacting control system.

Denoting by \( \mathbf{x} = [x, z] \in \mathbb{R}^4 \) the augmented state vector, the matrices involved in equations (2.3), (2.4) can be written

\[ \mathbf{A} = \begin{bmatrix} -2 & 0 & 0 & -1 \\ 0 & -2 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \] (5.3)

The subspaces

\[ \mathcal{P}_1 = \{ \mathbf{x} : x_2 = 0, x_3 = z \} \] (5.4)

\[ \mathcal{P}_2 = \{ \mathbf{x} : x_1 = 0, z = 0 \} \] (5.5)

are invariant under \( \mathbf{A} \).

Let \( \mathbf{B}_1, \mathbf{B}_2 \) be the \( 4 \times 1 \) submatrices corresponding to the column-wise partition of the matrix \( \mathbf{B} \). The subspaces \( \mathcal{P}_1, \mathcal{P}_2 \) are clearly the least invariant under \( \mathbf{A} \) containing \( \mathcal{P}(\mathbf{B}_1), \mathcal{P}(\mathbf{B}_2) \) respectively, so that the necessary and sufficient condition (2.5) is satisfied with the equality sign for the output subspaces

\[ \mathcal{Y}_1 = \{ y : y_2 = 0 \} \] (5.6)

\[ \mathcal{Y}_2 = \{ y : y_1 = 0 \} \] (5.7)

6. Hierarchic non-interaction

Often the subspaces \( \mathcal{Y}_1, \ldots, \mathcal{Y}_p \) correspond each to a particular set of output coordinates, so that they are mutually orthogonal. In such a case, an hierarchic non-interaction problem could be stated in the following way: determine an asymptotically stable control system, having the structure shown in Fig. 1, such that for every \( i \) (\( i = 1, \ldots, p \)) the input \( r_i \) does not affect the output on the subspaces \( \mathcal{Y}_1, \ldots, \mathcal{Y}_{i-1} \) and possibly completely controls the output on \( \mathcal{Y}_i \).
In other words, controls on \( \mathcal{Y}_1, \ldots, \mathcal{Y}_p \) are allowed in a hierarchical priority. By means of the synthesis method outlined above, the stated problem can be easily solved.

In order to describe the procedure, let consider a particular case. Only three subspaces \( \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3 \) mutually orthogonal and such that \( \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3 = \mathbb{R}^n \) are considered in the given hierarchical order and the corresponding \( \mathcal{Y}^* _1, \mathcal{Y}^* _2, \mathcal{Y}^* _3 \), together with their state space associated \( \mathcal{F}^* _1, \mathcal{F}^* _2, \mathcal{F}^* _3 \) are to be found.

To solve the problem, first determine, by means of relationships (3.1-3.3), the greatest output subspace \( \mathcal{Y}^* _1 \) contained in \( \mathcal{Y}_1 \) completely reachable by trajectories belonging to it. If the condition \( \mathcal{Y}^* _1 = \mathcal{Y} _1 \) is not satisfied, apply the same relationship to the subspaces \( \mathcal{Y}_1 + \mathcal{Y}_2, \mathcal{Y}_1 + \mathcal{Y}_3, \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3 \) in the given order, until a subspace containing \( \mathcal{Y} _1 \) is found, which is assumed to be \( \mathcal{Y}^* _1 \). Of course, the corresponding \( \mathcal{F}^* _1 \) is contemporary found.

The same procedure is then applied to the subspace \( \mathcal{Y}_2 \) in order to determine \( \mathcal{Y}^* _2 \) and \( \mathcal{F}^* _2 \); a first trial is made with \( \mathcal{Y} _2 \), and after, if it is unsuccessful, a second one is made with \( \mathcal{Y}_2 + \mathcal{Y}_3 \).

Finally, the subspace \( \mathcal{Y}_3 \) is considered and \( \mathcal{Y}^* _3, \mathcal{F}^* _3 \) are derived.

7. Conclusions

A procedure for the synthesis of non-interacting controllers in the state space has been described.

Necessary and sufficient conditions for non-interaction or decoupling have been given, which are very simply checkable on numerical computers, using standard routines for the calculus of matrices.

An asymptotically stable non-interacting control system can be always synthesized by means of the outlined procedure. An important related question, the minimization of the state space or, more generally, of the cost of the controller, is still open and could provide an interesting area for future investigations.

APPENDIX

Notational conventions

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<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>( \mathbb{R}^n )</td>
<td>the vector space of all ( n )-tuples of real numbers;</td>
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<tr>
<td>( A )</td>
<td>a matrix;</td>
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<tr>
<td>( A^* )</td>
<td>the pseudoinverse of the matrix ( A );</td>
</tr>
<tr>
<td>( \mathcal{F} )</td>
<td>a subspace;</td>
</tr>
<tr>
<td>( \mathcal{F}(A) )</td>
<td>the range of the linear transformation expressed by the matrix ( A );</td>
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<tr>
<td>( \mathcal{N}(A) )</td>
<td>the null-space of the linear transformation expressed by the matrix ( A );</td>
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the subspace of all vectors which are mapped in $\mathcal{F}$ by the linear transformation $A$, namely $A^{\ast\ast} \mathcal{F} = A^* (\mathcal{F} \cap R(A)) + \mathcal{F}(A)$; the least invariant with respect to $A$ containing $\mathcal{F}$; the greatest invariant with respect to $A$ contained in $\mathcal{F}$; the least subspace $\mathcal{J}$ containing $\mathcal{F}$ which satisfies the relationship $A (\mathcal{J} \cap \mathcal{F}) \subset \mathcal{J}$; the greatest $(A, \mathcal{F})$-controlled invariant contained in $\mathcal{F}$, i.e. the greatest subspace $\mathcal{J}$ contained in $\mathcal{F}$ which satisfies the relationship $A \mathcal{J} \subset \mathcal{J} + \mathcal{F}$; end of discussion.

REFERENCES


